Degree Structures and Their Finite Substructures

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  $$\mathcal{S}_r = (\mathcal{S}/{\equiv_r}, \leq)$$

  for a (usually countable) subfamily $\mathcal{S} \subset \mathcal{P}(\omega)$.

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All these lead to global (and many local) degree structures.

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- $\mathcal{D}$ is an upper semilattice (but usually not a lattice), i.e., $\mathcal{D}$ has a join operation $\deg(A) \cup \deg(B) = \deg(A \oplus B)$, where $A \oplus B = \{2x \mid x \in A\} \cup \{2x + 1 \mid x \in B\}$.

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- Most global degree structures support a "jump" operation $\mathbf{a} \mapsto \mathbf{a}'$ such that $\mathbf{a} < \mathbf{a}'$, and $\mathbf{a} \leq \mathbf{b}$ implies $\mathbf{a}' \leq \mathbf{b}'$.

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degree structure	complexity: 1st or 2nd order arithmetic	∃- or ∀∃-fragment decidable	∃∀∃-fragment undecidable
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If a class $\mathcal{C}$ of finite structures is ∃-definable with parameters in a degree structure $\mathcal{D}$, and the common ∀∃∀-theory of $\mathcal{C}$ is hereditarily undecidable, then the ∃∀∃-theory of $\mathcal{D}$ is undecidable.

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For the enumeration degrees, one can also code all finite distributive lattices as intervals (Lempp, Slaman, M. Soskova 2021).

Deciding the $\forall\exists$-theory of $\mathcal{D}$ amounts to giving a uniform decision procedure to the following

Problem (for deciding the $\forall\exists$-theory of $\mathcal{D}$)

Given finite partial orders $\mathcal{P}$ and $\mathcal{Q}_i \supseteq \mathcal{P}$ (for $i < n$), does every embedding of $\mathcal{P}$ into $\mathcal{D}$ extend to an embedding of $\mathcal{Q}_i$ into $\mathcal{D}$ for some $i < n$ (where i may depend on the embedding of $\mathcal{P}$)?

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For the Δ^0_2-Turing degrees, embed $\mathcal{L}$ both as an initial segment; and also $\mathcal{L} - \{1\}$ as an initial segment, mapping 1 to $\mathbf{0}'_T$.

Two natural subproblems of the ∀∃-theory are the following:

Extension of Embeddings Problem

Given finite partial orders $\mathcal{P}$ and $\mathcal{Q} \supseteq \mathcal{P}$, does every embedding of $\mathcal{P}$ into $\mathcal{D}$ extend to an embedding of $\mathcal{Q}$ into $\mathcal{D}$?

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The EE problem is decidable for the c.e. Turing degrees (Slaman/Soare 2001), for the enumeration degrees (Lempp/Slaman/Soskova 2021), and for the Σ^0_2-enumeration degrees (Lempp/Slaman/Sorbi 2005).

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The LE problem remains open for the c.e. Turing degrees, but is decidable for the Σ^0_2-enumeration degrees and for the enumeration degrees (Lempp/Sorbi 2002: all finite lattices embed).

Given the difficulty of the overall problem of deciding the $\forall\exists$-theory of the enumeration degrees and of the Σ^0_2-enumeration degrees, we are currently concentrating on the following subproblem of the Extension of Embeddings Problem for the Σ^0_2-enumeration degrees:

1-Point Extensions of Antichains

Decide, given a finite antichain $\mathcal{P} = \{a_0, \ldots, a_n\}$ and 1-point extensions $\mathcal{Q}_S = \{a_0, \ldots, a_n, x_S\}$ and $\mathcal{Q}^T = \{a_0, \ldots, a_n, x^T\}$ for some *nonempty* subsets $S, T \subseteq \{0, \ldots, n\}$ (where $x_S < a_i$ iff $i \in S$; and $x^T > a_i$ iff $i \in T$), Given the difficulty of the overall problem of deciding the ∀∃-theory of the enumeration degrees and of the Σ_2^0-enumeration degrees, we are currently concentrating on the following subproblem of the Extension of Embeddings Problem for the Σ_2^0-enumeration degrees:

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(It is always possible to extend an embedding of a finite antichain $\mathcal{P}$ to an embedding of the antichain $\mathcal{Q}_\emptyset = \mathcal{Q}^\emptyset$.)

The context for our subproblem is the two following earlier results:

Theorem (Ahmad 1989 (cf. Ahmad, Lachlan 1998))

1. There is an *Ahmad pair* of Σ^0_2-enumeration degrees $(\mathbf{a}, \mathbf{b})$, i.e., there are incomparable degrees $\mathbf{a}$ and $\mathbf{b}$ such that any degree $\mathbf{v} < \mathbf{a}$ is $\leq \mathbf{b}$.

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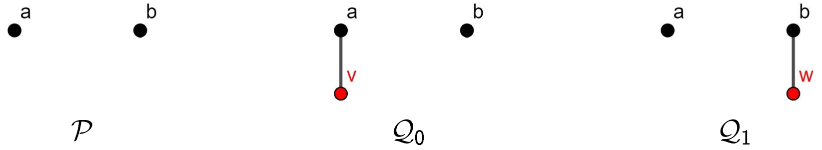
These are examples of ∀∃-statements blocking $\mathcal{P} \subset \mathcal{Q}_0$ but not $\mathcal{P} \subset \mathcal{Q}_0, \mathcal{Q}_1$:



We can handle the case of $\mathcal{Q}_{\mathcal{S}}$:

Theorem in Progress (Goh, Lempp, Ng, M. Soskova)

Fix $n > 1$ and $\mathcal{S} \subseteq \mathcal{P}(\{0, \dots, n\}) - \{\emptyset\}$.
Let $S_0 = \{i \leq n \mid \{i\} \in \mathcal{S}\}$, and let $S_1 = \{0, \dots, n\} - S_0$.

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Then some embedding of $\mathcal{P}$ into $\mathcal{D}_e(\leq \mathbf{0}'_e)$ cannot be extended to an embedding of $\mathcal{Q}_S$ for any $S \in \mathcal{S}$ iff

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   - for each $i \in S_0$, $\{i\} \cup \nu(i) \notin \mathcal{S}$, and
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The proof extends both results of Ahmad and combines them with minimal pair techniques.

As for $\mathcal{Q}^T$, we have to take into account the following

Theorem (Kalimullin, Lempp, Ng, Yamaleev 2022)

There is no cupping Ahmad pair, i.e., an Ahmad pair (a, b) with $\mathrm{a} \cup \mathrm{b} = \mathbf{0}'_e$.

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We conjecture that this is the only additional obstruction when considering extensions by points above an antichain:

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- $\mathcal{Q}_S$ satisfies the conditions of the Theorem in Progress, and
- any $T \in \mathcal{T}$ contains only one element, or contains two elements i, j with $j \in \nu(i)$ (from the Theorem in Progress).

Thanks!