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Figuring Out Plato's Divided Line

Darrell Dobbs

Wise rulers, according to Plato's *Republic*, are guided by insight into goodness itself; hence, the ἰδέα of the Good is the most important thing to learn (μέγιστον μάθημα, 505a2, 517b8-c5). But Socrates never tells his companions, even when pressed, what he thinks the Good is. Being told is not the same thing as learning. The aim of Socratic pedagogy is to facilitate eyewitness, not to extend the reach of hearsay. So, in the place of a doctrine, Socrates offers a succession of images—Sun, Line, and Cave—that listeners must do their own work to interpret. The central panel of Socrates' great triptych of images may be said to represent a peak example of his pedagogy, for even its formulation as an image would remain incomplete without the active participation of his interlocutor. Glaucon must subdivide each segment of an unequally cut line proportionately for the image of the Divided Line to be complete.

Glaucon's geometrical exercise of proportionate subdivision is as crucial to his own education as the mathematical studies Socrates prescribes later to his philosopher kings are to the dialectical journey that culminates in their vision of the Good (540a4-b7). In context, this geometric exercise serves to remediate the underlying deficiency responsible for Glaucon's failure to grasp Socrates' first image, the Sun. Glaucon rejected that image, ridiculing its culminating analogy as a mad exaggeration (δαιμονίας ὑπερβολῆς, 509b2-c2). So, a different image is needed. But this time Glaucon must acquire a personal stake in the image by collaborating in its completion. Through his own work of mathematical construction, Glaucon invests himself in the Divided Line image and simultaneously deepens his appreciation of proportionate reasoning and analogy.

But the importance of Glaucon's construction has been overlooked by commentators. No geometric reenactment of Glaucon's exercise is to be found anywhere in two millennia of extant scholarly literature. Indeed, there are many (albeit troublingly divergent) *illustrations*.[1] Yet there is not a single mathematically determined *construction*. In fact, it was not until the early twentieth century that a fundamental mathematical error in the received interpretation of the Line was corrected. Taking the Line to be the symbolic representation of cosmogenic emanations from an ultimate first principle, ancient, medieval, and modern readers treated it as a map of the learning soul's journey to enlightenment through contact with *continuously increasing* degrees of reality. The soul, on this account, reflected first on images and proceeded by means of the perception of

[1] See Smith 1996, 25-27 for a thorough accounting of the conflicting approaches taken to illustrating the Line by twentieth century commentators.

palpable but impure particulars; she then advanced further through consideration of pure though still multiple numbers and shapes—the so-called mathematical intermediates—and finally achieved genuine insight and understanding in the contemplation of the pure and singular forms (see Proclus 2022, 176-177). Manuscript scholia accordingly depicted the Line in four subsections, each one being greater in magnitude than the one preceding it (see Greene 1938, 246).

This doctrine so dominated scholarship that for more than two thousand years commentators failed to grasp the fact that Socrates' construction specifications for the Line necessarily entail *equality* of magnitude—and so, in some sense, equality of status—between the two types of cognition represented by its interior subdivisions. Whewell 1860, 444 arrived inductively at a conjecture concerning this equality. But it was Adam 1902, ii 64, who published a definitive algebraic proof. Despite this striking result, Adam remained personally convinced, and believed that Plato was convinced, that continuously increasing degrees of clarity were achieved in the soul's ascent through each of the four subsections of the Line. Adam tactfully muted the influence of his own seminal contribution, insisting that the equality of the Line's middle subsections represented 'a slight though unavoidable defect in the line, for [these sections are] not equal in point of clarity'.

Despite this courtesy, Adam's proof challenged Plato's mastery of metaphor. Subsequent scholars were less diplomatic. They openly maintained that Plato had failed to notice the parity implication inherent in his own composition.[2] Many remained unpersuaded of Plato's awareness of this parity, even in the wake of a compelling proof given 'in the Greek manner' by Klein 1965, 119.[3] Perhaps they felt, as Echterling 2018, 6 suggests, that any Greek of Plato's time who wished to prove the equality of the Line's middle subsections would 'almost certainly' rely on a more graphic approach. In any case, despite the availability of various demonstrations of the parity proposition,[4] the question of Plato's understanding or even awareness of its mathematical necessity remained a matter of controversy.

In recent years, however, commentators have come to believe that Plato must have known of the paradoxical parity. Leading scholars ingeniously suggest that he purposely planted inconsistency in Socrates' description of the Line to remind his reader of the imperfection of all imagery (e.g., Foley 2008, 18-19, 23, Smith 2019, 113-114, and Storey 2022, 11). Their desire to vindicate Plato is refreshing. But the specific arguments they produce to substantiate Socrates' or Plato's knowledge of parity beg the question. None of them shows how Socrates, within

[2] Storey 2022, 11 compiles a catalogue of commentators attributing this shortcoming to Plato.

[3] Klein's proof is Greek, to be sure. But it unfolds in the complete absence of formal construction and so does not provide the *demonstratio ad oculum*, which by Klein's own standards must be taken as a prerequisite of genuine geometry (Klein 1968, 6, 112; see also Knorr 1975, 69-73, Mueller 1981, 122, and Fowler 1987, 2, 110-113).

[4] See Ross 1951, 45, Brumbaugh 1952, 530, and Sinaiko 1965, 306 for additional twentieth century proofs.

the conversation he has with Glaucon, could have known that the interior subdivisions of the Line are equal in magnitude.[5] Hence, it is my intention here to show that cogent evidence of Socrates' (and *a fortiori* Plato's) awareness of this equality is, in fact, readily available. The discovery of this evidence simply requires that we take the educational drama of the *Republic* seriously enough to adopt the perspective shared by the interlocutors and perform the exact geometric construction Socrates assigns to Glaucon.

I. The Dramatic Figuration of Plato's Divided Line

Interpreting a Platonic dialogue requires sensitivity to the vitality of conversation. The best interpretation will account for all textual details while respecting the scope and limits of what the interlocutors themselves could be expected to make of the conversation as it unfolds.[6] The Divided Line passage expressly calls for such a reading, for it opens with Socrates urging Glaucon to remain mindful of the relevance of the preceding Sun analogy to the interpretation of the new image he is about to introduce.

> Keep in mind (νόησον), then, that the two [sc. the Good and the Sun] are, as we say, a pair and that the one reigns over the genus and domain of what is thinkable (νοητοῦ), while the other reigns over what is visible (ὁρατοῦ)… You do grasp these two kinds, the visible (ὁρατόν) and the thinkable (νοητόν)? …Well, take them then as a line divided into two unequal segments—one belonging to the class of what is seen (τοῦ ὁρωμένου) and the other belonging to what is thought (τοῦ νοουμένου)—and cut each segment again proportionately….[7] (509d1-8)

Though paired, the Good and the Sun do not rule over equal domains. So, Socrates' Line is divided into a greater segment and a lesser segment. This division between them is *complete* (τετμημένην, 509d6), suggesting that the Line's separation into regions representing what is 'seen, not thought' and what is 'thought, not seen' embodies the utter divorce or χωρισμός between the visible domain and the domain of the intelligibles (476a2-480a13, 507b9-10, 509d4). Consequently, it would be ill-advised to take the Line's integrity or coherence for granted, for to do so would trivialize the problem the Line is introduced to sym-

[5] Smith 1996, 42, Foley 2008, 14 and Storey 2022, 11 ground their claims on the putative equivalence of Socrates' two formulations of proportionalities (509d6-8, 510a8-10, cf. 533e7-534a5) but offer no explanation of the basis for Socrates' *knowledge* of this equivalence. Echterling's 2018, 11 diagrammatic argument derives the equality of the Line's interior subdivisions from the supposed, but not proven, equality of the alternate angles he labels η and γ.

[6] Kopff 1977, 113 cites Wilamowitz's precept that commentators on the comedies of Aristophanes should 'follow step by step the action that follows from the words…[showing] what that action would look like on stage'. I propose to apply the same to the task of interpreting the philosophical drama of Plato's Divided Line. For the general justification of this treatment of Plato's text, see Schleiermacher 1836, 14, 56-57, Strauss 1964, 52-60, Klein 1965, 4-5, and Gadamer 1980, 159.

[7] English translations of Plato are my own, rendering the Greek text as edited by Burnet 1902.

bolize. A solution to the problem of the χωρισμός, should there be one, must be *discovered* as the interlocutors' discussion of the Line image unfolds.

Nevertheless, presupposition of the Line's integrity is expressly advocated by Grube 1980, 28 and is implicit in any account that identifies a point of Golden Section as the *necessary* location of the Line's division (e.g., Brumbaugh 1954, 266 and 279, Pomeroy 1971, 389, Desjardins 1976, 492, Dreher 1990, and Olsen 2006, 6-7). Reading the Golden Section into the image implies that the Line *as given by Socrates* manifests a unifying continuous proportion—'the finest of bonds', as it is called by the Pythagorean Timaeus (*Timaeus* 31c2). I maintain that Socrates would find the claim of preestablished coherence to be facile, for it ignores the genuine perplexity of learning, i.e., of successfully bridging the gap between what we see and what we seek to understand.[8]

Just as the unequal division of the line is to be taken as a datum, so too is its verticality. This claim is hardly controversial, though the Line is frequently represented horizontally—most likely as a simple convenience of typography. There is, however, no need to jump ahead to Socrates' remarks later in the text (511a6-7; ἐπὶ τῷ ἀνωτάτω, 511d4), as is commonly done, to substantiate this verticality. Instead, *with Glaucon*, we may simply recall Socrates' earlier observation that the genuine philosopher, 'whose mind is truly oriented toward what is real (τοῖς οὖσι) has no leisure to look *downward* (κάτω) into the business of human beings' (500b8-c1). Indeed, the line Socrates presents is a straight-forward graphic representation of his own consistently articulated verbal metaphor: he draws a vertically oriented, unequally divided line, whose upper segment represents the domain of what is thinkable and whose lower segment represents the domain of what is visible.

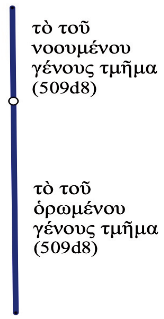
A question does remain, though, as to which of the two unequal segments of the line has the greater magnitude.[9] Isolated from dialogical context, the question is difficult to settle; one could justify assigning greater magnitude to either of the segments. But let us resolve this matter, as before, by determining what Glaucon may be expected to infer based upon Socrates' preceding statements. In the Sun image, Socrates repeated his customary distinction between the many beauties

[8] It is true, of course, that the Line *could* be divided at a point of Golden section. Even Balashov 1994, 285, perhaps the staunchest opponent of the Golden Section interpretation, stops short of arguing against that possibility. The point I am making is that if we were *to know* that the Line was cut at the Golden Section, the purpose of the Divided Line image in symbolizing the perplexity of the χωρισμός would be compromised. Hence, it is best to understand the Line's division as being unequal, precisely determined, yet nevertheless *unspecified*—just as the Greek indicates. Glaucon's construction of subdivisions must be accomplished accordingly, with sufficient *general* cogency to accommodate this lack of specificity.

[9] Plutarch 1976, 35-47 examines both possibilities before deciding, on grounds of the metaphysics of causal comprehensiveness, that the longer segment of the Line should be assigned to the intelligible realm. Two centuries later, Iamblichus 1975, 34-35 reports a view, attributed to a putatively ancient Pythagorean, Brontius, in favor of the visible realm occupying the longer segment—perhaps, as Brumbaugh 1954, 99 suggests, because 'the Pythagoreans tended to assign greater value to small rather than to large numbers'. Nevertheless, Proclus 2022, 177-178, two centuries after Iamblichus, canonized Plutarch's conclusion.

(πολλὰ καλά) and the beautiful itself (αὐτὸ δὴ καλόν, 507b2-10). Even earlier in the dialogue he had said that 'this applies to all the *eidê*, just, unjust, good, bad, namely, each in respect to itself is one (αὐτὸ μὲν ἓν ἕκαστον εἶναι), though because they come to be seen everywhere in association with bodies and actions and with one another, each appears to be many' (476a4-7). Socrates further maintains that the genuine philosopher, unlike the cosmopolitan sights-lovers who are commonly mistaken for philosophers, is always careful to distinguish the pure 'invisible look' (*eidos*), which is one, from its multiple phenomenal manifestations or participants (τὰ μετέχοντα, 475e9-476d3, 479b9-c5). Based on Socrates' previous statements, Glaucon will naturally expect that he intends the realm comprising these potent εἴδη to be represented by the more compact, less extensive segment of the Line.

So, in any faithful production of the drama of Plato's *Republic* Glaucon will be portrayed as accepting from Socrates a figure resembling the one depicted below.



The population of each of the segments of the diagram will be correctly apprehended by Glaucon based simply on Socrates' previous statements. But this diagram does not by any means complete the image; it is Glaucon who achieves this completion, in accordance with Socrates' prescription, by constructing proportionate subdivisions within each of the Line's original segments.

Socrates orders Glaucon to cut each segment again proportionately; this cut will produce in the segment assigned to what is seen (τῷ ὁρωμένῳ) different subdivisions relating to one another in respect of reliable clarity (σαφηνείᾳ) and obscurity (ἀσαφείᾳ).[10]

As he takes up this mathematical challenge, Glaucon can be expected to rely only on such geometrical principles as had come to be commonly accepted among educated Athenians of his time. Moreover, his solution must be achievable within the limited time made available as the conversation transpires. Accordingly, we do find that Socrates takes care to furnish time for the comple-

[10] As for the meaning of 'clarity', Glaucon will recall Socrates' saying a little earlier that 'when turned to things upon which the sun shines, the same eyes see clearly (σαφῶς) *and* sight is shown to be present in them' (508d1-2). So, σαφήνεια along the Divided Line is measured by an entity's capacity, in collaboration with some external source of illumination (ultimately the Good), to prompt insight into what it truly is. For some useful philological observations concerning the usage of σαφήνεια, see Lesher 2010, 180-181.

tion of the necessary construction. He does this by describing the anticipated subdivisions of the visible realm in terms that would appear to be, were they not serving this dramatic function, gratuitously verbose.

> In one subdivision you'll have images—by images I principally mean shadows, though also phantasms in water and all such apparitions constituted in dense, smooth, and bright surfaces, if you understand… The other subdivision you are to assign to that which these images are like, namely, the animals around us, all the plants, and the whole class of things made by art. (509d8-510a6)

In the time Socrates takes to provide these details, four strokes of Glaucon's stylus—provided one understands their motivation—will suffice to achieve the required subdivision. In fact, little more is required than a knack for applying Thales' famous sticks and shadows theorem.[11]

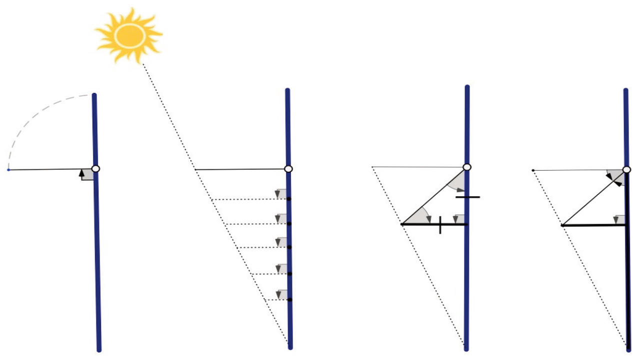
First, by erecting at the Line's original point of division a perpendicular 'stick' equal in length to the upper segment of the Line, Glaucon can reproduce the ratio between the Line's unequal segments in the ratio between this newly drawn perpendicular and the lower segment of the Line. Then, by completing the triangle enclosing this perpendicular and the lower segment, he can inscribe a third side comprising points from which additional perpendiculars can be dropped to the lower segment. Comprehending the proportionality implications of Thales' theorem, Glaucon and Socrates would understand that every one of these additional perpendiculars cuts the Line's lower segment such that the ratio between itself and the portion of the lower segment below its intersection with the Line is likewise *identical* to the original governing ratio of the Line. Nevertheless, only one of these many perpendiculars is equal in magnitude to the portion of the lower segment *above* its point of intersection. This would be the perpendicular, then, that cuts the lower segment *proportionately*, i.e., into upper and lower subdivisions governed by the same ratio governing the upper and lower segment of the Divided Line.

Because this perpendicular and the portion of the Line's lower segment above its point of intersection are equal, they form two legs of an isosceles right triangle. Glaucon and Socrates know that in an isosceles triangle the angles subtending equal sides are themselves equal.[12] So, each of its two remaining sub-

[11] Thales' feat in measuring the height of the Great Pyramid is recounted by Plutarch 1928, 147, who implicitly credits Thales with an appreciation of the proportionality of corresponding parts of similar triangles. But Diogenes Laertius i 27 reports an even earlier and more elaborate account indicating that Thales timed his measurement so that his stick's length was equal to the length of its own shadow, a maneuver that would not entail a general knowledge of proportionality. The latter, if it were indeed absent from the geometric knowledge of the sixth century figure of Thales, was provided by mathematicians in Ionia sometime between Thales' death and the flourishing of Hippocrates of Chios (Knorr 1975, 6-7), who composed his own pre-Euclidean book of geometry that included well-articulated principles of proportionality. Interestingly, this Hippocrates also happened to have resided for a considerable length of time in Athens sometime between 450 and 430 BCE (Heath 1921, i 183).

[12] Thales was credited by Aristotle's student, Eudemus of Rhodes, as well as by Plutarch and

tending angles must be ½ of a right angle. Therefore, to locate the sought-after perpendicular precisely, Glaucon need only construct the base of this isosceles right triangle. This he can do by bisecting the right angle contained between the Line's lower segment and the perpendicular he originally erected on the Line.



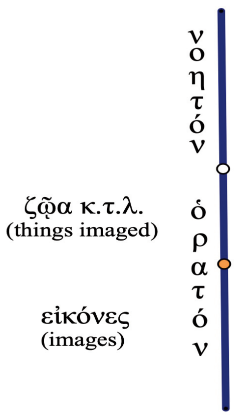
To comprehend the speed with which the steps described in this analysis can be executed, consider the figures above. So, in the time Socrates takes to differentiate and describe the populations of the two subdivisions of the lower division of the Line, Glaucon can sketch a solution to the geometry problem he has been assigned. In conversation, it would be tedious to provide the full apparatus of formal proof—though I shall provide the required demonstrations below in section 2. As my graphics above indicate, Glaucon could dramatically construct a fully cogent figure freestyle, as we might say today, 'on a cocktail napkin'.

Socrates has indicated that the relative magnitude of the subdivisions resulting from Glaucon's cut will reflect the degree of 'clarity and obscurity' of their respective populations. So, what are these populations and how do they differ? Socrates first mentions natural images, such as shadows and reflections in pools of water or in mirrors, and then, subsequently, all that which this [class of likenesses] is like (ᾧ τοῦτο ἔοικεν, 509e1-510a5). Socrates starts with entities defined by the darkness that results from an *obstruction* of illumination. A shadow's form is highly susceptible to change with even the slightest alteration in the relationship between that obstruction and the source of illumination. Shadows are 'shifty', and so are more obscure and less reliably clear than the objects they are like, especially when the latter are seen directly, i.e., face-to-face. Socrates identifies these shadows as *images* (εἰκόνες), insofar as they invite conjecture concerning that whose shadows they are. The reliability of the resulting cognition may not be great, but it is not nothing. Hence, we must distinguish the

Diogenes Laertius, with knowing the conditions for congruency (cf. Euclid I.4 and I.26), the angle-sum theorem (at least for right angles, cf. I.32), the equivalence of the angles subtending equal sides in isosceles triangles, as well as the converse of this proposition, including the capacity to bisect an angle (cf. I.5, 6, 9; see Gow 1884, 140-145 and Heath, citing Todhunter, in Euclid 1956, 258). But, again, even if Thales' knowledge is somehow exaggerated by the early sources, there is no doubt among historians of mathematics that all these principles were publicized in Hippocrates' *Elements* prior to the earliest posited dramatic date of the *Republic* (Knorr 1975, 6-7 and Verlinsky 2014).

cognitive experience of recognizing an image as an image from the cluelessness of idolatry, which we suffer when we mistake an image for what it images.

Because Glaucon's cut establishes within the segment of the visible the same ratio obtaining between the lower, more extensive visible domain and the higher, less extensive noetic domain, his cut will likewise yield unequal subdivisions: a longer subdivision situated *below* a shorter one. Socrates assigns the longer and bottommost subdivision of the Line to images, the visible entities *most lacking* in reliable clarity. He directs Glaucon to assign (τίθει, 510a5) the other subdivision of the Line to the class of things that these images are like.



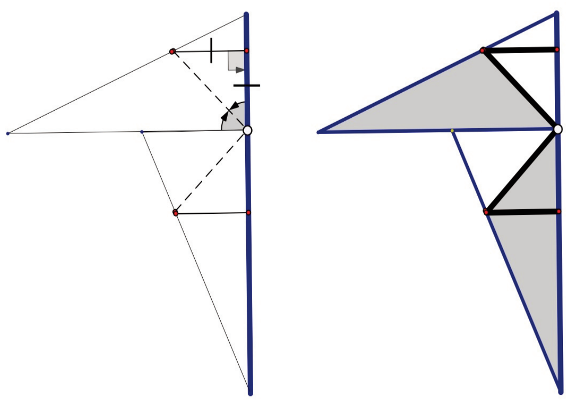
Difficulties arise at this point because readers tend to assume that Socrates intends that magnitude in the Line should vary *positively* with 'reliable clarity'.[13] But there is no textual support for this assumption. What Socrates says is that a proportionate division of the lower segment of the Line will yield different subdivisions 'relating [proportionately] to one another in reliable clarity and obscurity' (σαφηνείᾳ καὶ ἀσαφείᾳ πρὸς ἄλληλα, 509d9). The resulting correlation of magnitude and clarity could be either direct or inverse. But the evidence we have already culled from Socrates' preceding statements suggests that the association between magnitude and clarity on the Line is, indeed, *inverse*. Moreover, as Socrates works out the significance of the Line image, we find that clarity is enhanced by division, not by composition. The domain of visible things is clarified when Socrates distinguishes the respective populations of images and the things they image. The conflation of these divisions only increases obscurity, culminating in a condition resembling the catastrophic idolatry of the Cave, where images are not distinguished from what they image (515a5, c1-2). The experience of seeing an image as an image is represented by a subsegment shorter than the segment representing the entire composited visible realm. Evi-

[13] Among the commentators, only Grube 1980, 25; 1974, 164 consistently assigns the Line's shorter segment to the *noêta*—though unfortunately he offers no explanation for his attribution. Brumbaugh 1954, 99 offers a similar illustration, but reverses himself in 1989, 82. Reeve also straddles the issue, giving two opposing depictions of the Line (see 1988, frontispiece and 2004, 205).

dently, linear magnitude on the Divided Line varies inversely with reliable clarity.

Socrates next asks Glaucon if 'in regard to trueness (ἀληθείᾳ) and its opposite, images stand to what they image as the opinable (τὸ δοξαστόν) stands to the knowable (τὸ γνωστόν, 510a8-10).' His question associates the opinable with the likened (τὸ ὁμοιωθέν) and the knowable with that to which it is likened (τὸ ᾧ ὡμοιώθη). Glaucon's assent affirms, then, that from our own experience of images and what they image we can infer that our mediated conjectures (δόξαι) can be distinguished as being less 'true' than the acquaintanceship (γνῶσις) we achieve through an immediate encounter. So, we distinguish trueness and its opposite (ἀληθείᾳ τε καὶ μή, 510a9) by reflecting upon the presence *in the visible realm* of a differentiation analogous to Socrates' earlier differentiation of the visible realm as opinable (δοξαστόν) and the noetic realm as knowable (γνωστόν, 479d3-10). This analogy induces Glaucon to conclude that an entity's clarity, i.e., the degree to which it fosters reliable insight into what it is, is a function of its *genuineness* (ἀλήθεια). Like clarity, 'trueness,' is inversely correlated with magnitude in the Line image.

Glaucon meanwhile has time to repeat in the Line's upper segment a proportionate subdivision analogous to the one he has already achieved in its lower segment. Naturally, the second time working through a problem goes faster than the first. The repetition takes no more than a moment. The exactitude of likeness between the two exercises of subdivision obviously produces three pairs of congruous triangles. Both Socrates and Glaucon know that corresponding parts of congruous triangles are equal and that *the interior subdivisions of the Line are corresponding parts of congruous triangles*.



So, once Glaucon's sketch is completed, it is perfectly evident to both interlocutors that the middle subdivisions of the Divided Line are, by geometric necessity, equal in magnitude.

Of course, we must keep in mind that Socrates has not yet differentiated the populations of the resulting subdivisions of the upper segment of the Line. So, he urges Glaucon to *look again*, *closely*, at the point at which it was necessary to cut the noetic realm (σκόπει δὴ αὖ, 510b2). Glaucon's initial consideration focused on the technical problem of proportionate subdivision. The *closer examination* for which Socrates now calls involves the differentiation of resident populations rather than the mere determination of magnitude, which, as we may infer from the comparative adjective *closer*, Glaucon has already accomplished.

Socrates' call for closer examination also helps us to understand the oft-repeated observation that neither Socrates nor Glaucon mentions the equality of the Line's middle subdivisions. Those who take the view that Foley 2008, 9-12 calls 'demarcationist' regard this silence as highly significant in suggesting that Plato does not intend anything by this equality and may well himself have been unaware of it. I wish simply to point out that in the current dramatic circumstances there is no reason for the interlocutors to mention this equality. Because Glaucon's cut *precedes* the differentiation of populations within the noetic realm, nothing is yet specified within the noetic realm to be equated with that portion of the visible realm populated by that which images are like. Hence, nothing of interest about that equality can *yet* be said. This limitation holds for the duration of Socrates' discussion of the populations of the noetic realm—in other words, through the remainder of the discussion of the Divided Line in book 6. Later, though, we shall encounter an indirect Socratic reference to the equality of the Line's interior parts, when the obvious, though still unspoken, equality figures decisively in Socrates' alternative expression of the Line's proportionality. But it is important to note that only the geometric construction of the Line image as I have given it can provide a basis for Socrates' formulation of the new expression of proportionality at 533e7-534a5.

Nevertheless, even prior to Socrates' differentiation of the specific populations, one might reasonably ask how any merely opinable thing can be equated with *any* knowable thing in terms of trueness or clarity. This is indeed a perplexity, but blaming Plato with contradicting himself does not help resolve the difficulty. Still, owing to this difficulty, many learned commentators have been led to contend that the Divided Line cannot even be drawn.[14] Instead let us keep in mind that reliable clarity, as Klein 1985, 289-293 explains, is not the only measure of epistemic excellence. Some quality other than clarity may account for whatever superiority may be attributed to our cognition of certain noetic entities over our cognition of the palpable things in our environment. For example, it appears to be the case that cognitions informed by the population of the lower subdivision of the noetic realm are superior *in precision or exactitude* to cognitions informed by the population of the higher subdivision of the visible realm. But as we consider this possibility it is important to keep in mind that superiority in precision does not imply superiority in cognitive clarity, i.e., *in facilitating*

[14] Socrates twice ranks *dianoia* above *pistis* (511d8e2, 533e7-534a1). To some, this ranking seems to be at odds with the parity of the Line's interior subdivisions. Ross 1951, 44-45, Brumbaugh 1954, 98, Sinaiko 1965, 165, and Sallis 1975, 415 expressly claim that this conflict makes it impossible to make an accurate drawing of the Line. But kudos to Storey 2022 for refusing to take the inconsistency for granted.

insight into what each thing truly is.

As he considers the population of the lower division of the noetic realm, Socrates identifies a type of thinking facilitated but also constrained by its hypotheses. Thinking of this sort originates as investigators perfect or idealize *wholly in their minds* that which is given imperfectly in visible forms.[15] Such idealizations yield not only 'the square itself' of Socrates' geometer, but also the physicist's perfect vacuum and the economist's notion of perfect competition—each of which provides a conceptual basis for a deductive descent to conclusions that can be grasped only by discursive thinking (διάνοια, 511a1).[16] I suggest that it is only these hypothetical idealizations and their implications—not the visible forms consulted to suggest them—that belong to the rational structures constituting the 'objects' of *dianoia* (*pace* Smith 1996, et al.).

Now, the marginal increase in precision achieved by these structures of reason does not by itself produce any deepening of insight into what is ultimately true or genuine. One can gain precision without gaining wisdom. Beautiful and ugly, Socrates says, 'tumble about' within the visible domain; yet even when *dianoia* is summoned to sort them out, one still falls short—pending *dialectical* clarification—of grasping the ultimate significance of each (479d3-10, 523a5-524d6). Indeed, the deepest insight requires even more than a glimpse of the beautiful itself. One must see that beauty partakes of the Good. How might one go about catching a glimpse of this greatest learning matter? The parity of the Line's interior subdivisions suggests that to achieve this end one might just as well contemplate the loving works of Francis of Assisi as Thomas Aquinas's more precise arguments for God's existence. On the same basis, it is reasonable to expect that Abraham Lincoln, pondering the battle of Gettysburg, can achieve as deep an understanding of *what war is* as John Nash can by puzzling out the theorems of non-cooperative game theory.

The equality of the interior subdivisions of the Line may also carry additional significance: it is possible that the parity of their populations in providing access to 'what is' may shed light on what Wigner 1960 famously calls the 'unreasonable effectiveness of mathematics in the natural sciences'. As I have indicated, Socrates is able to present alternative formulations of the proportional relationships present on the Divided Line thanks to the demonstrated equality of the interior subdivisions of the Line (cf. 5096-8 and 511d6-e4 with 533e7-534a5). But this parity is not attributed by Socrates to any causal relationship between them,

[15] Gonzalez 1998, 231-234 maintains that this power of 'idealization' extends beyond the construction of mathematical hypotheses, playing a part in the philosopher's dialectical interrogation, which advances toward the *eidê* themselves.

[16] Any objection that these conceptual constructs lack 'reality' is as toothless in discussing Plato's Divided Line as it is in discussions of physical or economic theory (see Duhem 1991, 39 and Friedman 1953, 3). Yet, to claim, with the traditional advocates of mathematical intermediates, that Socrates held such constructs *to exist* outside the mind may well place an unbearable semantic burden on the shoulders of the Greek verb, *einai* (see Kahn 1966, 255-257). Let us remember, instead, that the status Socrates claims for these constructs, as well as their associated theorems, is that of a dream (533b6-c3). Importantly, though, such dreams can foster an attachment to the truth (572a5-b1).

much less to any exactitude of number.[17] Parity simply represents the fact, elaborated in Socrates' two formulations of proportionality, that the populations of each subsection *bear the same relation to some same third thing*, viz., an *eidos*, the very thing that each one really is (αὐτὸ ἓν ἕκαστον εἶναι, 476a5-6). Intelligibility attaches to the relationship between the objects of *dianoia* and the objects of *pistis* owing to the symmetry of their relationships to the *eidê*. It is owing to the symmetry of these relationships, perhaps, that it is possible to make accurate predictions about 'the animals around us' based simply on our well-formulated scientific theories.

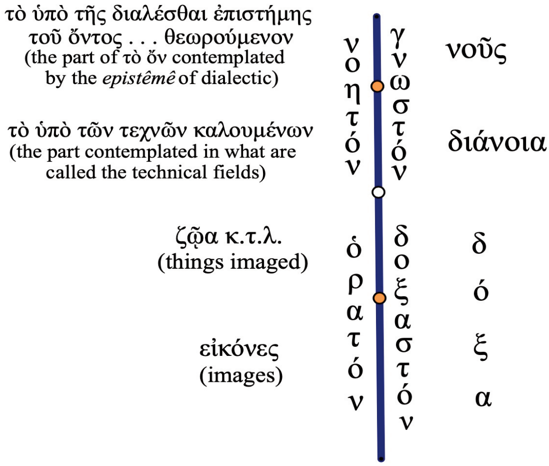
Moving on to the highest level of the Line, Socrates identifies a kind of thinking more advanced than the mathematical. This thinking is launched by an hypothesis, but it is propelled up and beyond the realm of the hypothetical (ἀνωτέρω ἐκβαίνειν, 511a6 and b6-7). The thinking thus launched is dialectical and scientific rather than merely technical. It ascends upward toward the unhypothesized beginning of all (τὴν τοῦ παντὸς ἀρχήν, 510b4-511c2). From there it descends to a conclusion without depending at all on perception, making its way entirely, as Socrates says, through *eidê* and ending in *eidê*.

With Socrates' differentiation of these two modes of thinking as a preface (τούτων προειρημένων, 510c1-511c2), Glaucon is prepared to state his tentative understanding of the populations of the noetic realm. He takes it that Socrates makes his distinction on the basis that

> the part of the essential and noetic realm contemplated by the science of dialectic is clearer than the part contemplated by investigators in the so-called technical fields, where hypothetical starting points are required to view these things with thought rather than perception. But because the examination conducted by these investigators does not ascend to an authoritative beginning and is instead directed deductively from hypotheses, they do not seem to you to achieve a fully insightful understanding (νοῦς) of these things, though they do still count as noetic, i.e., thinkable, because they are, after all, joined with a rational first principle (καίτοι νοητῶν ὄντων μετὰ ἀρχῆς, 511c4-d4; see Denniston 1981, 559).

[17] That ὅρατα outnumber νόητα is uncontroversial (476a2-7). Likewise, in the visible realm, images outnumber things imaged and, in the noetic realm, intangible equilateral triangles, e.g., outnumber the eidetic essence of triangularity. But it is hard to arrive at a comparative numerical estimate of the domains represented by the *interior subsections* of the Line. One might attempt to argue, nevertheless, that insofar as the objects of διάνοια are generalizations they must be counted as being *fewer* in number than what they generalize. But let us not overlook that διάνοια comprehends not only an untold number of intangible equilateral triangles; it also comprehends an untold number of possible sentences that can be truly affirmed about those triangles—as well as about all the parts of those triangles and everything composed of those triangles. Furthermore, any attempt to count the possible parts and composites, present, past and future, of all the 'animals, plants, and artifacts around us' is subject to the same difficulty. Even supposing that Glaucon were required, which he is not, to think of the parity of the interior subdivisions of the Line in terms of number rather than in terms of trueness and clarity (509d9, 511e1-4), he might well respond that the two are indeed equal—equally untold.

The inferior part of the noetic realm, the one contemplated by practitioners in the so-called technical fields (ὑπὸ τῶν τεχνῶν καλουμένων, 511c6), includes, as I suggest, the entire conceptual apparatus of axioms, models, theorems, and laws familiar to us in what we call scientific theory.[18] Though these rational structures do not allow for the deepest or most penetrating insights, they have been constituted as a coherent and rationally discernible whole, beginning with a hypothetical principle of their own. Hence, they may be said to be knowable (γνωστόν) and are rightly located within the noetic realm. Yet, in consideration of their deficiencies, Glaucon assigns them to the lower, longer and less reliably clear subdivision of the noetic realm, while he assigns the invisible *eidê* contemplated by dialectic to the shorter, clearer, and more elevated subdivision.



Before concluding the report on Socrates' differentiation of the noetic realm, Glaucon makes one final observation. Socrates, he surmises, calls the cognitive bearing of geometers and others like them '*dianoia* rather than *nous* because *dianoia* is, in some sense, between *doxa* and *nous*' (μεταξύ τι δόξης τε καὶ νοῦ, 511d2-5). The 'sense' (τι) in which *dianoia* is between *doxa* and *nous* refers, of course, to their relative degree of clarity as embodied in the magnitudes of the Divided Line.[19] Glaucon takes Socrates to be coining a *special* usage for *dianoia*, the general Greek word for thinking. The prefix *dia* suggests a passage from one

[18] For detailed accounts of the rational structure of natural and social scientific theory, see Duhem 1991, Friedman 1953, Hempel 1966, 70-84, and Rudner 1966, 10-53; for illustrations of specific theories, consider Newton 1934, Stigler 1952, and Riker 1962.

[19] Glaucon's 'betweenness' could be taken to refer merely to the *location* of the subsegment representing *dianoia*. That, however, would miss the point, for it is linear magnitude that conveys the metaphorical significance of the qualities Socrates would have Glaucon keep foremost in mind (ἡγησάμενος, 511ee2-4). If Glaucon were to fix his attention on location rather than magnitude, his literal-mindedness would preclude any appreciation of Socrates' instruction regarding the summoning power of images. But that result is incompatible with Socrates attestation to the adequacy (ἱκανώτατα, 511d6) of Glaucon's reception of his account.

thing to another, so *dianoia* seems an apt word for thinking though the confusions endemic to *doxa* but without yet achieving the more fully insightful understanding of *nous*.[20] Glaucon, as we have noted, speculates on this point of language while *looking directly* at the Line he has helped Socrates to construct. By visually confirming the 'in-between' clarity of *dianoia*, Glaucon corroborates the *inverse* association of clarity and magnitude on the Line. Indeed, with the help of this remark it is possible for us *to demonstrate* that διάνοια can be 'between' νοῦς and δόξα in magnitude only on the condition that the segment representing the visible domain is indeed greater than the segment representing the noetic domain.

Socrates responds directly to his companion's summation, saying that Glaucon's take-away is quite sufficient (ἱκανώτατα…ἀπεδέξω, 511d6). He does, however, add some specific labels—terms of art, so to speak—for the cognitive experiences that arise in the soul in association with the populations he and Glaucon have differentiated. In correspondence with the different types of visible things, Socrates identifies within *doxa,* or conjecture, an experience of *eikasia*, which we may describe as an image-inspired awareness of visible entities around us, and an experience of *pistis*, which is the confident awareness of an eyewitness (511d6-e4). There is also a reality beyond that which our eyes see. This greater reality, as the equality of the interior subdivisions of the Line already implies, cannot consist merely of the rational structures of geometry and kindred arts contemplated by *dianoia*, but includes the more elevated part of the noetic realm disclosed by the science (ἐπιστήμη) of dialectic. Socrates provisionally labels the experience associated with this realm *noêsis*. After benefitting from a more extensive discussion of dialectic later in the dialogue, he applies the more sharply focused term, *epistêmê* (533e8). In concluding his discussion of the Line image, Socrates directs Glaucon to arrange these four cognitive experiences, along with that which incites each, in proportion to their trueness and reliable clarity. Thanks to his collaboration in constructing and interpreting the Divided Line, Glaucon now finds Socrates' references to proportionality and analogy illuminating rather than bedazzling, as he had earlier in in the case of the Sun Analogy. This time he responds strikingly with an unreserved μανθάνω ('I understand').

II. Formal Proofs Required to 'Figure Out' the Divided Line

Here now are four proofs that will confirm my previous analyses and will suffice to 'figure out' the *Republic*'s Divided Line: [21]

Proposition 1

Given a line divided into two unequal sections, to subdivide each section pro-

[20] Socrates has not yet introduced his own terms (νόησις, πίστις, and εἰκασία) for the other three παθήματα that populate the Line. It is his mention of διάνοια at 511a1 that provokes Glaucon's remark.

[21] Here I adopt Euclid's structure of synthetic proof, though I eschew his use of the perfect passive imperative. For a discussion of the possible implications of this syntactical usage, see Lachterman 1989, 61-67.

portionately.

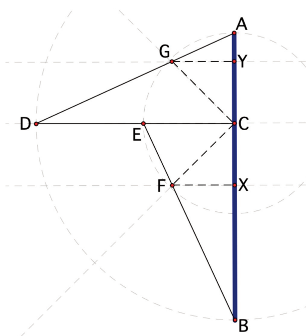
Let AB be that line, with C lying anywhere between A and B on AB except its midpoint. It is required to cut CB at X and AC at Y such that AC:CB :: CX:XB and AC:CB :: AY:YC.

Construction:

1. With Center C, describe Circle A with radius CA and Circle B with radius BC.
2. Through C erect a line perpendicular to AB, intersecting Circle A at E and Circle B at D on the same side of AB; connect CE, CD, AD, and BE.
3. Bisect ⊥BCE and ⊥ACD, intersecting BE at F and AD at G, respectively.
4. Through F and G draw lines perpendicular to AB, intersecting AB at X and at Y respectively. Wherever on AB Point C is located, I say that AC:CB :: CX:XB and AC:CB :: AY:YC.

Proof:

1. EC:CB :: FX:XB Thales' 'sticks and shadows' Theorem (cf. Euclid, *El.* VI.4).



2. ∠XCF =∠CFX ⊥FXC is right by construction; ∠XCF is half of bisected ⊥BCE by construction, so ∠CFX is half a right angle by angle sum of ΔCFX (I.9, 32).
3. FX=CX In a triangle, the sides subtended by equal angles are equal (I.6).
4. AC:CB :: CX:XB Substituting equals: equal radii, AC for EC, and equal sides, CX for FX.
5. AC:CD :: AY:YG Thales' 'sticks and shadows' Theorem (VI.4).
6. ∠YGC=∠YCG ⊥GYC is right and ∠YCG is half of bisected ⊥ACD, by construction, so ∠YGC is half a right angle by the angle sum of ΔCGY (I.9, 32).
7. YG=YC In a triangle, the sides subtended by equal angles are equal (I.6).
8. AC:CB :: AY:YC Substituting equals: equal radii, CB for CD, and equal sides, YC for YG. ὅπερ ἔδει ποιῆσαι

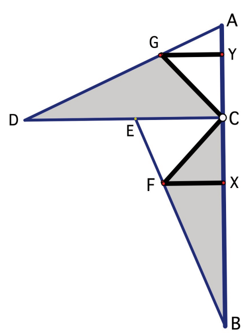
Note that this construction produces perfectly proportionate subdivisions in an already unequally divided line *wherever* that original division happens to be located. The generality of my proof, which accommodates that ‘wherever’, respects the mysterious nature and provenance of the split between the noetic and visible realms. Wherever the Line is originally divided, Glaucon’s cuts precisely embody Socrates’ mandated proportionalities and render the Line, in its entirety, rationally intelligible. If the given Divided Line may be said in some sense to represent the perplexity of the human situation, the proportionate response provided by Glaucon’s execution of Socrates’ prescription has disclosed this situation to be one that is hospitable to an inquisitive rational soul.

Now, looking more closely at the construction, we cannot fail to notice three pairs of congruent triangles. All three pairs arise by mathematical necessity from the operations conducted in accordance with Socrates’ explicit directions. The necessary congruency of these triangles, and *therefore the equality of their corresponding parts*—including the middle subsections of the original Line—is therefore inescapable.

Proposition 2

If each of the two segments of an unequally divided line are cut proportionately, the smaller part of the larger segment is equal to the larger part of the smaller segment.

Highlighting the figure from Proposition 1 to emphasize congruencies, I say that XC=CY.



Proof:

1. ΔBCE ≅ ΔDCA SAS (I.4): CB=CD (equal radii of Circle B, I def.15, 16); ⊥BCE =⊥DCA (I post.4); AC=CE (equal radii of Circle A, I def.15, 16).
2. ΔBCF ≅ ΔDCG ASA (I.26): ∠CBE=∠CDA (corresponding parts of ΔBCE ≅ ΔDCA, I.4); BC=CD (equal radii of Circle B, I def.15, 16); ∠BCF=∠DCG, each being a bisection of a right angle (I post.4).
3. ΔCXF ≅ ΔYCG ASA (I.26): ∠XCF=∠YCG (each being a bisection of a right angle, I post.4); CF=CG (corresponding parts of congruent triangles ΔBCF ≅ ΔDCG, I.4); ∠CFX=∠CGY (each being half a right angle, Proposition 1steps 2 and 6).

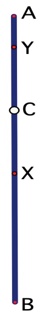
4. XC=CY Corresponding sides of congruent triangles, ΔXCF ≅ ΔYCG (I.4). ὅπερ ἔδει δεῖξαι

By making Socrates' simple instructions his starting point and completing the mandated construction, even the proverbial Greek schoolboy would immediately see the truth of the parity proposition. After demonstrating that such insight can arise from an elementary geometrical construction, no basis remains for supposing that Glaucon or Socrates or Plato could be unaware of the parity proposition. Scholars, for more than a century, have questioned whether other valid proofs of this proposition truly suffice to demonstrate this awareness. In view of the anachronistic or abstruse character of those proofs, their questions were not entirely unreasonable. But, by reenacting Glaucon's role and following Socrates' express instructions, I have shown that a little geometry can settle the matter.

We have already had occasion to discuss the brief passage, later in book 7, in which Socrates offers a reformulation of the proportions among the parts of the Divided Line (533a4-5). Socrates suggests that the two formulations are indeed equivalent (ὥσπερ τὸ πρότερον, 533e7). But on what basis can he do so? The answer is simpler than previous scholars have suggested. There is no need to invoke and justify the scope and use of the operation of *alternando*. Because the equality of the middle subdivisions of the Line is immediately obvious from the congruencies constructed and highlighted in Propositions 1 and 2, the mathematical justification of the equivalence of Socrates' two formulations of proportionality requires nothing more than the straightforward substitution of equal magnitudes.

But the ease of this substitution should not disguise the profundity of what it signifies. Socrates' affirmation of the equivalence of the two sets of proportions indicates that our cognition of rational mathematical theorems *and* our direct empirical cognition of the world around us stand in precisely the same relation to the eidetic reality disclosed by the ἐπιστήμη of dialectic. This sameness of relationship can be traced to the differentiation of segments of the Line accomplished in proportionate imitation of the Good's own proportionate begetting of the Sun. The gymnastics of mathematical thinking yield hypotheses and conclusions that constitute the elements of scientific theory. These theories achieve an amazing degree of correlation with patterns observable in the phenomena of the visible domain. Theorems embedded within these theories are indeed likenesses of the genuine εἴδη. But anyone who mistakes them for the εἴδη, to say nothing of the Good itself, is merely dreaming. By neglecting the dialectical interrogation of these theories and celebrating only their predictive power in relation to the phenomena of the visible realm, one diverts and enfeebles the upward impulse and erotic drive empowering the quest for wisdom. Though Socrates' principal interest was to advocate, model, and engage in this dialectical quest for wisdom, he was nevertheless able to foresee in passing, as it were, a foundation for the possibility of mathematical physics.

A formal proof of the equivalence of Socrates' two formulations of proportions is, by this point, anticlimactic.



PROPOSITION 3

When each segment of a line divided unequally in two is cut proportionately, the smaller parts of each segment also stand to one another in that same ratio – as do the larger parts of each segment.

In Line AB, where AC:CB :: CX:XB :: AY:YC, I say that AC:CB :: AY:CX :: YC:XB.

Proof:

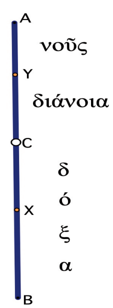
1. AC:CB :: AY:YC :: CX:XB proportions of the Line from Proposition 1.
2. AC:CB :: AY:CX :: YC:XB substitution of equals, XC=CY, from Proposition 2. ὅπερ ἔδει δεῖξαι

Finally, let us recall that Glaucon takes Socrates' *dianoia* to designate a cognitive experience 'between' *doxa* and *nous* in reliable clarity (511d3-5). I will now show that this observation, together with other mathematical data given in the text, suffice to demonstrate the logical impossibility, and hence the absurdity, of the prevalent view that holds that Socrates intends to represent the noetic realm with the longer original segment of the Line.

PROPOSITION 4

In an unequally divided line subdivided proportionately, if the magnitude of a subdivision of one segment of the line is between that of the other subdivision of the same segment and the entire remaining segment, the magnitude of that remaining segment is greater than that of the segment comprising the aforementioned subdivisions.

In Line, AB, if YC is between AY and CB in magnitude I say that CB is greater than AC.



Proof:
Suppose to the contrary that the upper, noetic segment of the Line is longer than its lower segment. By proportionate subdivision, it follows that νοῦς occupies the longer and διάνοια the shorter subdivision of the longer segment, so νοῦς > διάνοια; and because διάνοια is between νοῦς and δόξα in magnitude, it follows that νοῦς > διάνοια > δόξα, so διάνοια > δόξα. Of course, πίστις is a part of δόξα and it is axiomatic that the whole is greater than the part, so, δόξα > πίστις. Nevertheless, because we know (from Proposition 2) that διάνοια = πίστις, it necessarily follows that δόξα > διάνοια! So, simply by supposing that the upper segment of the Line is longer than the lower segment we are led to contradict ourselves, saying both that διάνοια > δόξα and that δόξα > διάνοια. Therefore, we are obliged to reject the supposition of the noetic segment's greater magnitude. We must instead conclude that the upper, noetic realm occupies the *shorter* segment of the unequally Divided Line. It follows necessarily that magnitude on the Divided Line correlates *inversely* with reliable clarity. ὅπερ ἔδει δεῖξαι

While linear magnitude along the *Republic*'s Divided Line is indeed a function of reliable clarity, it is an *inverse* function. The segment of the Line corresponding to the intelligible realm must be smaller than the segment corresponding to the visible realm, and so on with respect to the clarity of the proportionately divided subdivisions of each of the Line's original segments.

Conclusion

I have interpreted the image of the Divided Line in real dramatic time, intentionally deferring to the unfolding of the conversation between Socrates and his interlocutor rather than to some preconceived cosmological doctrine. Doing so, however, imposes a unique burden. A key element in the drama of Plato's *Republic* consists in Glaucon's successful construction of proportionate subdivisions of an unequally divided line. A cogent dramatic interpretation must account *rigorously* for the mathematical details of this construction. This is a challenge previous commentators have never undertaken. Its successful completion is apodictically verifiable. I have produced a mathematical analysis and formal geometric proofs—and even an indirect argument by contradiction—to corroborate fully the details of an interpretation formulated with sensitivity to the conversation's transpiring in real dramatic time. Of course, there remains much in this account that calls for further thought and discussion, but I hope to have established a secure foundation for further inquiry.

It appears that Socrates, too, upon completing his discussion of the Divided Line, believes that such inquiry is warranted; indeed, he had anticipated from the outset that the presentation of the Line would leave many things out (συχνά γε ἀπολείπω, 509c7). So, to fill in these lacunae he introduces a third image. Glaucon is to liken the condition of our nature with respect to its education and aversion to education to the experience of prisoners bound from childhood in a cave-like dwelling. Described in the starkest terms, human nature—apart from external, forceful intervention—tends to oscillate between complacent idolatry and befuddled dazzlement. This feature of the human condition was 'left out' of the sunnier Line image, where the commensuration of genuine (albeit graded) experiences of cognition, vouchsafed by the Good, provided a basis for confidence in the possibility of learning and so suggested the hospitableness of the cosmic situation to a thinking intellect. But the Cave Allegory, attached as a supplement to the brilliant (but failed) Sun analogy and the pedagogically optimistic Divided Line, exposes the radical dependency of human well-being on the real presence of a transcendent Good. Apart from the order and illumination provided by this Good, anything—from the lowliest shadow to the purest idea of justice—can reduce our minds to base idolatry. The Cave's disarming disclosure of our fundamental need for this Good is not meant to counsel despair but to ignite and fuel a longing for contact with this Good. Though one can hardly claim here to have attained such contact, we have through Socrates' three images perhaps caught an indirect glimpse of this greatest learning matter as that which is most needful if we are ever to know what is real and to do what is right.[22]

Department of Political Science
Marquette University
Milwaukee WI 53233

BIBLIOGRAPHY

Adam, James ed. 1963. *The Republic of Plato.* 2 vols. Cambridge: Cambridge University Press.

Balashov, Yuri. 1994. 'Should Plato's Line Be Divided in the Mean and Extreme Ratio?' *Ancient Philosophy* 14: 283-295.

Brumbaugh, Robert. 1952. 'Plato's Divided Line' *Review of Metaphysics* 5: 529-534.

Brumbaugh, Robert. 1954. *Plato's Mathematical Imagination*. Bloomington: Indiana University Press.

Brumbaugh, Robert. 1989. *Platonic Studies of Greek Philosophy.* Albany: SUNY Press.

Burnet, John. 1902. *Platonis Res Publica*. Oxford: Clarendon Press.

Denniston, J.D. 1981. *The Greek Particles*. Oxford: Clarendon Press.

Desjardins, Gregory. 1976. 'How to Divide the Divided Line' *Revue of Metaphysics* 29: 483-496.

Diogenes Laertius. 1980. *Lives of Eminent Philosophers*. R.D. Hicks trans. Cambridge: Harvard University Press.

Dreher, John P. 1990. 'The Driving Ratio in Plato's Divided Line' *Ancient Philosophy* 10: 159-172.

Duhem, Pierre. 1954. *The Aim and Structure of Physical Theory*. Princeton: Princeton University Press.

Echterling, Terry. 2018. 'What Did Glaucon Draw?' *Journal of the History of Philosophy* 56: 1-15.

Euclid. 1926. *The Thirteen Books of Euclid's Elements*. Thomas L. Heath ed. and trans. 3 vols. Cambridge: Cambridge University Press.

Foley, Richard. 2008. 'Plato's Undividable Line: Contradiction and Method in *Republic* VI' *Journal of the History of Philosophy* 46: 1-24.

Fowler, David H. 1987. *The Mathematics of Plato's Academy*. Oxford: Clarendon Press.

Friedman, Milton. 1953. *Essays in Positive Economics*. Chicago: University of Chicago Press.

Gadamer, Hans-Georg. 1980. *Dialogue and Dialectic*. New Haven: Yale University Press.

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Gonzalez, Francisco J. 1998. *Dialectic and Dialogue: Plato's Practice of Philosophical Inquiry*. Evanston: Northwestern University Press.

Gow, James. 1884. *Greek Mathematics*. Cambridge: Cambridge University Press.

Greene, William C. 1938. *Scholia Platonica*. Haverford PA: American Philological Society.

Grube, George Maximilien Antoine. 1980. *Plato's Thought*. Indianapolis: Hackett.

Heath, Thomas L. 1921. *A History of Greek Mathematics*. 2 vols. Oxford: Clarendon Press.

Heath, Thomas L. 1956. *The Thirteen Books of Euclid's Elements*. vol 1. New York: Dover.

Hempel, Carl G. 1966. *Philosophy of Natural Science*. Englewood Cliffs NJ: Prentice-Hall.

Kahn, Charles. 1966. 'The Greek Verb "To Be" and the Concept of Being' *Foundations of Language* 2: 245-265.

Klein, Jacob. 1965. A C*ommentary on Plato's Meno.* Chapel Hill: University of North Carolina Press.

Klein, Jacob. 1968. *Greek Mathematical Thought and the Origin of Algebra.* Cambridge: MIT Press.

Klein, Jacob. 1985. 'On Precision' 289-308 in Robert B. Williamson and Elliot Zuckerman edd. *Lectures and Essays*. Annapolis: St. John's College Press.

Knorr, Wilbur Richard. 1975. *The Evolution of the Euclidean Elements*. Boston: D. Reidel.

Knorr, Wilbur Richard. 1993. *The Ancient Tradition of Geometric Problems*. New York: Dover.

Kopff, E. Christian. 1977. 'Nubes 1493ff: Was Socrates' Murdered?' *Greek, Roman, and Byzantine Studies* 18: 113-122.

Lachterman, David Rapport. 1989. *The Ethics of Geometry*. London: Routledge.

Lesher, James H. 2010. 'The Meaning of "Saphêneia" in Plato's Divided Line' 171-187 in Mark L. McPherran ed. *Plato's Republic: A Critical Guide.* New York: Cambridge University Press.

Mueller, Ian. 1981. *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*. New York: Dover.

Newton, Isaac. 1934. *Mathematical Principles of Natural Philosophy. Vol.1: The Motion of Bodies*. Andrew Motte trans. Berkeley: University of California Press.

Olsen, Scott. 2006. *The Golden Section: Nature's Greatest Secret*. New York: Bloomsbury.

Plutarch. 1928. *Moralia* vol. II. '*Dinner with Seven Sages*'. Frank Cole Babbitt trans. Cambridge: Harvard University Press.

Plutarch. 1976. *Moralia* vol. XIII, part 1. '*Platonic Questions*'. Harold Cherniss trans. Cambridge: Harvard University Press.

Pomeroy, Sarah. 1971. 'Optics and the Line in Plato's *Republic*' *Classical Quarterly* 21: 390-392.

Proclus. 2022. *A Commentary on Plato's Republic. Vol II*. Dirk Baltzey, John F. Finamore, and Graeme Miles edd. and trans. Cambridge: Cambridge University Press.

Reeve, C.D.C. 1988. *Philosopher-Kings: The Argument of Plato's Republic*. Princeton: Princeton University Press.

Reeve, C.D.C. trans. 2004. *Plato's Republic*. Indianapolis: Hackett.

Riker, William H. 1962. *The Theory of Political Coalitions*. New Haven: Yale University Press.

Ross, David. 1951. *Plato's Theory of Ideas.* Oxford: Oxford University Press.

Rudner, Richard S. 1966. *Philosophy of Social Science.* Englewood Cliffs NJ: Prentice-Hall.

Sallis, John. 1975. *Being and Logos*. Bloomington: Indiana University Press.

Schleiermacher, Friedrich. 1836. *Introductions to the Dialogues of Plato*. William Dobson trans. London: John Parker.

Sinaiko, Herman L. 1965. *Love, Knowledge, and Discourse in Plato*. Chicago: University of Chicago Press.

Smith, Nicholas. 1996. 'Plato's Divided Line' *Ancient Philosophy* 16: 25-46.

Smith, Nicholas. 2019. *Summoning Knowledge in Plato's Republic*. Oxford: Oxford University Press.

Stigler, George. 1952. *The Theory of Price*. New York: Macmillan.

Storey, Damien. 2022. '*Dianoia* and Plato's Divided Line' *Phronesis*. 68: 1-56.

Strauss, Leo. 1964. *City and Man*. Chicago: University of Chicago Press.

Verlinsky, Alexander. 2014. 'Lysias' Chronology and the Dramatic Date of Plato's *Republic*' *Hyperboreus* 20: 158-198.

Wigner, Eugene. 1960. 'The Unreasonable Effectiveness of Mathematics in the Natural Sciences' *Communications in Pure and Applied Mathematics* 13: 1-14.